Delay Analysis of a Time Division Multiple Access (TDMA) Channel

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Outline

- What is TDMA
- Why analyzing the delay of TDMA
- How to do
- How about non-preemptive priority queue discipline
- Numerical Examples
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Definition of TDMA

- Short for *Time Division Multiple Access*, a technology for delivering digital wireless service using time-division multiplexing (TDM).
- TDMA works by dividing a radio frequency into time slots and then allocating slots to multiple calls. In this way, a single frequency can support multiple, simultaneous data channels.

Consider a TDMA system in which time is slotted and time slots are organized into frames of M slots, indexed from 1 to M.

- Time slots with the same index in consecutive frames form a TDMA channel.

- Let the duration of a frame be T seconds, a station using TDMA will transmit one packet a data into a time slot of \( \frac{T}{M} \) seconds and then become idle for \( (M-1)\frac{T}{M} \) seconds.
TDMA Channels

[Diagram showing frames labeled with time indices and time intervals marked with 'T SECONDS.']
What is TDMA

Why analyzing the delay of TDMA

How to do

How about non-preemptive priority queue discipline

Numerical Examples
Synchronous time division multiplexing was previously studied by Chu and Konheim [2].

Their model permits a general distribution for the number of packet arrivals within a time slot and they solved for the probability generating function of the queue size at time instant just prior to the beginning of a time slot.

They also obtained the expected delay experienced by a “virtual” message arrival.
By assuming Poisson message arrivals and employing a different analytic approach, we obtained different results for the steady-state probability generating function of the queue size as seen by a random observer as well as the expected delay actually experienced by messages.
Outline

- What is TDMA
- Why analyzing the delay of TDMA
- **How to do**
  - How about non-preemptive priority queue discipline
- Numerical Examples
Some Assumptions

- A station with a fixed assigned TDMA channel and unlimited buffer capacity for queueing is considered.
- Assume that data message arrive at the station according to an independent Poisson process.
- For each message, one or more fixed-length packets are formed.
Some Assumptions (cont’d)

- The number of packets comprising a message is given by a general probability distribution.
Terminologies

- **Service time (S)**
  - The time during which the packet is at the head of the queue with no transmission in progress and its transmission time of $T/M$ seconds.
  - It is a random variable ($X$) distributed between $T/M$ and $T+T/M$.
  - The service time of a message is the aggregate service time of its constituent packets.
Terminologies (cont’d)

- **Waiting time (W)**
  - The duration from a message enters the queue until the first packet be served.

- **Idle period (I)**
  - I is exponentially distributed under the assumption of Poisson arrivals.

- **The random variable Y**
  - Defined to be X – T/M.
  - Function of I.
Terminologies (cont’d)

- $N_t$
  - The number of messages in the system at time $t$.
  - Both in queue and service.
  - We want to find the steady-state probability generating function of $N_t$. 
Illustration of Busy and Idle Periods, Waiting and Service Times
The Queueing System

- A single-server queue with Poisson arrivals at $\lambda$ messages per second.
- The service time distribution of a message which initiates a busy period is $\hat{B}^*(s)$ with the first and second moment $\hat{b}_1$ and $\hat{b}_2$.
- All subsequent messages in the same busy period have service times drawn independently from the distribution $B^*(s)$ with first and second moments $b_1$ and $b_2$. 
Define the Transforms

\[ P_n = \lim_{t \to \infty} \text{Prob} [N_t = n] \]

\[ P(z) = \sum_{n=0}^{\infty} z^n P_n \]

\[ B^*(s) = \int_{0}^{\infty} e^{-sx} \, dB(x) \]

\[ \hat{B}^*(s) = \int_{0}^{\infty} e^{-sx} \, \hat{B}(x) \]
Theorem 1 (Welch [6])

- If $\lambda b_1 < 1$, then

$$P(z) = \frac{P_0[z\hat{B}^*(\lambda - \lambda z) - B^*(\lambda - \lambda z)]}{z - B^*(\lambda - \lambda z)} \quad (1)$$

- Where

$$P_0 = \frac{1 - \lambda b_1}{1 - \lambda(b_1 - \hat{b}_1)} \quad (2)$$

Recall...
Consider Packet Number in a Message

Let the number of packets in a message be $L$ which is a random variable given by the probability density $\{g_i\}_{i=1}^{\infty}$ with first and second moments $L_1$ and $L_2$ where

$$g_i = \text{Prob} [L = i]$$
Consider Packet Number in a Message

- The message service time distributions are given by the following transforms

\[
\hat{B^*}(s) = \sum_{l=1}^{\infty} g_l [\beta^*(s)]^l \quad \hat{B^*}(s) = \hat{\beta}^*(s) \sum_{l=1}^{\infty} g_l [\beta^*(s)]^{l-1}
\]

- Where

\[
\beta^*(s) = e^{-sT} \quad \hat{\beta}^*(s) = e^{-st/M} \int_0^T e^{-sy} \, dF(y)
\]

\[F(y) = \text{Prob} [ Y \leq y]\]
Obtaining $Y$

\[ F(y) = \begin{cases} 
  \frac{e^{-\lambda(T - \frac{T}{M})} [e^{\lambda y} - 1]}{1 - e^{-\lambda T}} + U \left( y - T + \frac{T}{M} \right) 
  & \text{if } 0 \leq y \leq T 
  \\
  1 
  & \text{if } y > T
\end{cases} \]

Where

\[ U(x) \triangleq \begin{cases} 
  1 & x \geq 0 \\
  0 & x < 0
\end{cases} \]
The first and second moment of $X$ are thus given by

$$
\overline{Y} = \left( T - \frac{T}{M} - \frac{1}{\lambda} \right) + \frac{T e^{-\lambda(T - \frac{T}{M})}}{1 - e^{-\lambda T}} \quad (3)
$$

$$
\overline{Y^2} = \left( T - \frac{T}{M} - \frac{1}{\lambda} \right)^2 + \left( \frac{1}{\lambda} \right)^2 + \left( T^2 - 2 \frac{T}{\lambda} \right) \frac{e^{-\lambda(T - \frac{T}{M})}}{1 - e^{-\lambda T}} \quad (4)
$$
Obtaining $X$

- The first and second moment of $X$ are thus given by

$$\bar{X} = \bar{Y} + \frac{T}{M} \quad (5)$$

$$\bar{X}^2 = \bar{Y}^2 + 2 \frac{T}{M} \bar{Y} + \left(\frac{T}{M}\right)^2 \quad (6)$$
The first and second moments of message service time are

\[ b_1 = L_1 T \]  \hspace{1cm} (7)

\[ \hat{b}_1 = (L_1 - 1)T + \overline{X} \]  \hspace{1cm} (8)

\[ b_2 = L_2 T^2 \]  \hspace{1cm} (9)

\[ \hat{b}_2 = \overline{X^2} + 2\overline{X}(L_1 - 1)T + (L_2 - 2L_1 + 1)T^2 \]  \hspace{1cm} (10)
Rewrite $P_0$ (cont’d)

- From equation (2) (3) (5) (7) (8)

\[
P_0 = \frac{1 - \lambda b_1}{1 - \lambda (b_1 - \hat{b}_1)}
\]

\[
= \frac{(1 - \lambda L_1 T)(1 - e^{-\lambda T})}{\lambda T e^{-\lambda (T - T/M)}}
\]

(11)
Expected Number of Messages

- Derived equation (1) at $z=1$

\[
\bar{N} = \frac{\lambda \hat{b}_1}{1 - \lambda (b_1 - \hat{b}_1)} + \frac{\lambda^2 (\hat{b}_2 - b_2)}{2[1 - \lambda (b_1 - \hat{b}_1)]}
\]

\[
+ \frac{\lambda^2 b_2}{2(1 - \lambda b_1)}.
\]

(12)
Expected Message Delay

- By applying Little’s formula

\[
\bar{D} = \frac{\hat{b}_1}{1 - \lambda(b_1 - \hat{b}_1)} + \frac{\lambda(\hat{b}_2 - b_2)}{2[1 - \lambda(b_1 - \hat{b}_1)]} + \frac{\lambda b_2}{2(1 - \lambda b_1)}.
\] 

(13)
Expected Service Time of a Message

\[ S = (1 - P_0)b_1 + P_0 \hat{b}_1 = b_1 - P_0(b_1 - \hat{b}_1) \]

\[ = \frac{\hat{b}_1}{1 - \lambda(b_1 - \hat{b}_1)} \quad (14) \]

- Hence, the expected waiting time is

\[ \tilde{W} = \frac{\lambda(\hat{b}_2 - b_2)}{2[1 - \lambda(b_1 - \hat{b}_1)]} + \frac{\lambda b_2}{2(1 - \lambda b_1)} \quad (15) \]
Rewrite the Formula

\begin{align}
\bar{N} &= \lambda L_1 T - \frac{\lambda T}{2} + \frac{\lambda T}{M} + \frac{\lambda^2 L_2 T^2}{2(1 - \lambda L_1 T)} \\
\bar{D} &= L_1 T - \frac{T}{2} + \frac{T}{M} + \frac{\lambda L_2 T^2}{2(1 - \lambda L_1 T)} \\
\bar{S} &= \frac{1}{\lambda} + \left( \frac{L_1 T - \frac{1}{\lambda}}{\lambda} \right) \left( 1 - e^{-\lambda T} \right) \\
\bar{W} &= L_1 T - \frac{T}{2} + \frac{T}{M} + \frac{\lambda L_2 T^2}{2(1 - \lambda L_1 T)} - \frac{1}{\lambda} \\
\quad - \left( \frac{L_1 T - \frac{1}{\lambda}}{\lambda} \right) \left( 1 - e^{-\lambda T} \right) \\
\quad \left( \lambda T e^{-\lambda (T - T/M)} \right). 
\end{align}
Suppose burst transmission rate is $C$ bits/s.

So, the data rate of a TDMA channel is thus $C/M$ bits/s.

Consider a FDMA channel which transmits continuously at $C/M$ bits/s.

The expected message delay for such a channel is given by the Pollaczek-Khinchin (P-K) formula.
Compare with FDMA (cont’d)

- Recall P-K formula

\[
W = \frac{\lambda E[X^2]}{2(1 - \rho)}
\]

- Thus

\[
\bar{D}_{PK} = L_1 T + \frac{\lambda L_2 T^2}{2(1 - \lambda L_1 T)}
\]
The traffic intensity is

\[ \rho \triangleq \lambda L_1 T \]

Thus, the expected message delay for a TDMA channel can be expressed in terms of the expected message delay for a FDMA

\[ \bar{D} = \bar{D}_{PK} - \frac{T}{2} + \frac{T}{M} \]

(20)
Observation

- If the duration $T$ of a time frame is shrunk to zero while the distribution of message lengths remains fixed, we have

\[
\lim_{T \to 0} \bar{D} = \bar{D}_{PK}
\]
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Some Definitions

- Consider the same system as in the previous section.
- Let there be $K$ message priority classes indexed from 1 to $K$ where 1 denotes the highest priority level.
- Message in the $k^{th}$ priority class are assumed to arrive at $\lambda_k$ messages per second.
- Define

$$\lambda = \sum_{k=1}^{K} \lambda_k$$
Some Definitions (cont’d)

- The number of packets in a class k message is given by the probability density $\{g_l^{(k)}\}_{l=1}^{\infty}$ with first and second moments $L_1^{(k)}$ and $L_2^{(k)}$.

- Define $\rho_k = \lambda_k L_1^{(k)} T$

- $b_1^{(k)}$, $\hat{b}_1^{(k)}$, $b_2^{(k)}$ and $\hat{b}_2^{(k)}$ are defined as in the previous section. Let

$$b_1 = \sum_{k=1}^{K} \frac{\lambda_k}{\lambda} b_1^{(k)} \quad \hat{b}_1 = \sum_{k=1}^{K} \frac{\lambda_k}{\lambda} b_1^{(k)}$$
Theorem 2

- If \( \sum_{i=1}^{K} \rho_i < 1 \), then the expected waiting time of a class \( k \) message is

\[
\bar{W}_k = \frac{V}{\left(1 - \sum_{i=1}^{k-1} \rho_i\right)\left(1 - \sum_{i=1}^{k} \rho_i\right)} \quad k = 1, 2, \ldots, K
\]  

(21)

- Where

\[
V = \frac{P_0}{2} \sum_{k=1}^{K} \lambda_k \hat{b}_2^{(k)} + \frac{1 - P_0}{2} \sum_{k=1}^{K} \lambda_k b_2^{(k)}
\]  

(22)

Consider SMF Priority Discipline

- The expected waiting time of a message of \( l \) packets is

\[
\bar{W}_l = \frac{V}{\left(1 - \lambda \sum_{i=1}^{l-1} i g_i T\right) \left(1 - \lambda \sum_{i=1}^{l} i g_i T\right)}
\]  

(26)

- Where

\[
V = \frac{\lambda P_0}{2} b_2 + \frac{\lambda (1 - P_0)}{2} b_2
\]  

(27)
Expected Service Time

The expected service time of a message of $l$ packets is

$$\bar{S}_l = P_0[(l - 1)T + \bar{X}] + (1 - P_0)lT$$

$$= lT - P_0(T - \bar{X}).$$
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Expected Message Delay Versus Traffic Intensity

- The burst transmission rate be $C$ bits/s.
- $C = 1.5$ Mbits/s
- $M$ (number of Slots) = 450.
- $T$ (duration of a frame) = 0.3 seconds.
- Each packet size is 1000 bits.
- Message is consists of either single-packet or eight-packets with $g_1 = \alpha$ and $g_8 = 1 - \alpha$.
- Three cases are shown for $\alpha = 1, \frac{8}{9}, \frac{1}{2}$
Result

![Diagram showing expected message delay versus traffic intensity for TDMA and FDMA systems with parameters C=1.5 MBPS, M=450, T=0.3 sec. The graph plots traffic intensity (ρ) on the x-axis and expected message delay (sec) on the y-axis. Different lines represent different values of α (α=1/2, α=8/9, α=1) for TDMA (solid line) and FDMA (dashed line).]
Expected Message Delay Versus Frame Time

- Each message is 8000 bits long.
- $C = 1.5 \text{ Mbits/s}$.
- $M = 300$
- $T = 1.6$ seconds.
- Each message can be transmitted as a single packet.
Result

Expected Message Delay (sec) vs Frame Time (sec)

C = 1.5 MBPS
M = 300
8000 Bits/Msg

\( \rho = 0.7 \)

\( \rho = 0.5 \)

\( \rho = 0.3 \)

\( \rho = 0.1 \)
Expected Message Delay Versus Traffic Intensity for SMF Queue Disciplines

- C = 1.5 Mbits/s.
- M = 450.
- T = 0.3 second.
- Packet size 1000 bits.
- Message length distribution, $g_1 = 0.3$ and $g_l = 0.1$ for $l = 2, 3, \ldots, 8$. 
Result

C = 1.5 MBPS
M = 450
T = 0.3 sec

EXPECTED MESSAGE DELAY (sec)

TRAFFIC INTENSITY $\rho$

TDMA
FDMA
Expected Message Delay Versus Message Length for SMF Queue Discipline

- The expected message delay increases with the message length.
- This is often a desirable feature since most applications typically require a much smaller delay constraint for short messages (interactive data traffic) than long messages (batch data traffic).
Result

![Graph showing the expected message delay vs. message length. The graph includes lines for different values of \( \rho \): 0.2, 0.4, 0.6, and 0.8. The parameters given are:

- \( C = 1.5 \) MBPS
- \( M = 450 \)
- \( T = 0.3 \) sec]
Recall

- In M/G/1 queue, the z-transform of probability distribution function is

\[ \Pi(z) = \frac{p_0(1-z)K(z)}{K(z)-z} \]

- Where

\[ K(z) = B^* [\lambda(1-z)] \]